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Recent Developments in Determination of the Lunar Gravitational Field from Satellite Orbits

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Recent developments in determination of the lunar gravitational field from satellite orbits

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The initial determinations of the variations in the lunar gravitational field are appreciably milder than those of the Earth in the sense of stress-implication, indicating a state closer to hydrostatic equilibrium. The variations determined also have a considerable correlation with the lunar topography, indicating a shallower origin than the Earth's variations.

The data are still insufficient to determine firmly the lunar oblateness, and thus help resolve the problem of the Moon's moment of inertia.

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1. Introduction

The satellites which are now in orbit around the Moon comprise two principal projects: (1) the U.S.S.R. Luna satellites, which have semi-major axes of about 1·39 lunar radii, eccentricities of about 0·13 and inclinations to the Moon's equator of about 72°; and (2) the U.S.A. Lunar Orbiter satellites, which have semi-major axes of about 1·54 lunar radii, eccentricities about 0·33, and inclinations about 12°. Two satellites of each series have attained selenocentric orbit, and results from the analysis of one orbit of each series are now available.

2. Description of the orbits and their analysis

Using estimates based upon: (1) the second degree components obtained from lunar motion and physical libration (Jeffreys 1961; Eckert 1965); (2) the visible topography plus the assumption of uniform density (Goudas 1966); and (3) the application of the hypothesis that the variations in the Moon's gravitational field reflect the same shearing stresses as those in the Earth (Kaula 1963), we arrive at various estimates for the variations of the Moon's gravitational field as expressed by the normalized harmonic coefficients $\bar{C}_{lm}$, $\bar{S}_{lm}$ of the potential from $\pm 10^{-4}/l^2$ to $\pm 2\times 10^{-4}/l$. The relatively slow rotation of the Moon with respect to inertial space (27·33 days) and of the pericentre of a lunar satellite (on the order of 400 days), coupled with these magnitudes of the coefficients, results in long-period perturbations by zonal harmonics (such as J_3) of probably some tens of kilometres amplitude, and by several tesseral harmonics of more than one kilometre amplitude (Kaula 1967). There should even be appreciable perturbations of a short-periodic nature. The aforestated magnitudes would lead to short periodic perturbations amounting to about ± 500 m position and ± 50 cm/s velocity.

Comparing these perturbations to the accuracy of the tracking systems of about ± 5 m range and $\pm 0{\cdot}1$ cm/s range rate, there appears to exist a rich complex of information. Furthermore, the capability of tracking the satellite almost continuously, all the time except when it is occulted by the Moon, is a luxury to which those who have worked with Earth satellites are unaccustomed. There is the difficulty, of course, that at any instant only two components of the satellite state are strongly observable: position and velocity in the direction from the Earth, the former of which may be entangled with inadequacies of the lunar ephemeris. In the short run some additional information is obtainable by using two tracking stations on the Earth. In the long run almost complete information is obtainable as the Earth revolves around the Moon with a period of 27 days with respect to inertial space. So we should expect that from a single satellite orbit the principal data analysis difficulty remaining after 1 month's tracking would be the ambiguity in separating different perturbations of similar periodicities: in particular, separating different variations of the lunar gravitational field. The only complete solution to this problem is the same for lunar satellites as for Earth satellites: a variety of orbital specifications.

The complexity of the problem is such that a variety of approaches is warranted.

The effort at N.A.S.A.-Langley Research Centre headed by W. H. Michael has the purpose of determining the gravitational field in order to plan orbital specifications for the primary photographic mission, but also is designed to obtain the variations of the gravitational field for scientific purposes (Michael & Tolson 1967). The approach adopted is that of numerically integrating the orbits in rectangular coordinates, 12th order double-precision, and of forming partial derivatives of the ranges and Doppler counts themselves with respect to the parameters being determined. Rather large and elaborate computer programs have been developed for this purpose, taking into account not only variations of the gravitational field, but also errors in lunar ephemeris, possible effects of the outgassing from the attitude control system of the satellite, radiation pressure, errors in the tracking station position on Earth, etc. This effort is supported by a group from Computer Usage Corporation headed by C. Sheffield, who also worked with the Earth satellite tracking effort at the U.S. Naval Weapons Laboratory (U.S.N.W.L.), Dahlgren, Virginia (Anderle 1966). The approach is indeed the closest to that of the U.S.N.W.L. of the various Earth satellite analysis efforts.

The approach adopted by Lorell (1967) of the Jet Propulsion Laboratory (J.P.L.), Pasadena, is almost at the opposite extreme of the possibilities: only the long-period variations of the orbit will be utilized, and the observations will be compressed to Keplerian elements from day to day. The analysis applied by Lorell will be of the long-term variations of the Keplerian elements over the month with the short-periodic variations averaged out. In this averaging, interactions between short-period terms which might give rise to long-period effects are neglected.

In the analysis of Kaula (1967), computer programs designed for Earth satellite orbits have been adapted to lunar satellite orbits. This approach utilizes a quasi-analytic theory in which the short-periodic variations are averaged out analytically taking into account the second-order interactions between short-period terms giving rise to long-periodic effects in forming an averaged disturbing function. This averaged disturbing function is then used for a numerical integration of the long-period intermediary. Second-order terms in the disturbing function thus have coefficients including products or squares of the Earth's mass, and the J_{20}, J_{22} and J_{30} of the Moon's field. The observation equations are formed in terms of range rate; the range rates to be used will probably be compressed over several minutes from those which are used in the analyses using orbits numerically integrated in rectangular coordinates. The short-periodic perturbations which must be included in perturbations at the instant of observation or in forming the partial derivatives are obtained analytically as periodic linear forced oscillations.

The details of the analysis of the U.S.S.R. Luna-10 orbit have not yet been published. Akim (1966) states that an analytic theory encompassing a 2-month interval was used, but that the motion was 'described' in rectangular coordinates.

The Luna-10, launched on 31 March 1966, was tracked in orbit around the Moon from 3 April to 30 May (Akim 1966). There is no indication of any difficulties affecting tracking accuracy. A second selenocentric orbit was attained by Luna-11, launched 24 August. The results of the Luna-10 orbit analysis are given in table 1.

The history of the Lunar Orbiter I satellite, which was launched 13 August 1966, can, from the selenodetic point of view, be divided into three phases:

(1) The first phase was 14 to 21 August, during which the Lunar Orbiter had a relatively high pericentre and moderate eccentricity of 0·30, an inclination of 12°. During this phase the attitude control was not operated, the satellite was left strictly undisturbed and no telemetry was taken. The major effort was to obtain as much tracking by the J.P.L. Deep Space Instrumentation Facility (D.S.I.F.) as possible, to analyse this tracking to determine the perturbations, and thence to make a decision as to how far the pericentre could be brought down toward the lunar surface for the purpose of taking photographs. The analysis was performed by two entirely distinct programs, both of them utilizing numerical integration. Rather consistent results were obtained solving on the order of about ten to twenty tesseral harmonics (Michael, Tolson & Gapcynski 1966). The consequence of only about 5 days tracking was that assurance was given rather confidently that the pericentre of Lunar Orbiter I could be brought down to within about 40 or 50 km of the lunar surface.

(2) The second phase was from 21 August to 16 September. During this period the photographs of the lunar surface were being taken, the attitude control was operated (although usually at rather wide intervals of the time, on the order of more than daily intervals). Furthermore, during this phase the tracking of the lunar satellite was frequently less accurate because the same carrier used for tracking was used for telemetering, with the consequence that the modification of the shape of the wavetrain by the telemetry often caused a mismatch of the return signal with the reference signal. Analyses carried out during this phase obtained results for the gravitational variations differing appreciably from those obtained during the first phase. The maximum residuals remaining after these analyses were on the order of 1 cm/s, usually in the vicinity of pericentre, where the satellite is moving most rapidly. This size residual is unacceptably large compared to the tracking accuracy; however, it is small compared to what we would expect to be the effect of the gravitational variations not included in the model, and hence the erratic results obtained for these harmonic coefficients can perhaps be attributed to some extent to the fact that they are undoubtedly soaking up considerable higher-order effects, as well as suffering from contamination of the tracking signal by telemetry.

(3) Phase three was from 16 September to the deliberate crashing of the orbiter in early November. The photography by Lunar Orbiter I was completed 16 September, after which the satellite was left in the same orbit, without operation of the attitude control

system, and with extremely limited telemetry. The satellite was tracked two orbits per day by the D.S.I.F. system. Consequently this month should produce the optimum information, both because of the sensitivity of the orbit to perturbations, and the amount of good tracking data. It is too early as yet to give much significant results on the analysis of tracking for this phase.

3. Initial results

The first results obtained from analysis of 52 days of the Luna-10 orbit have been published by Akim (1966). Eleven spherical harmonic coefficients of the lunar gravitational field were obtained. Both a second degree (C_{20}) and fourth degree (C_{40}) coefficient are included, but it is stated that their correlation coefficient was -0.99. However, no other correlation coefficient was greater than 0·4.

Michael, Tolson & Gapcynski (1966) solved for 21 coefficients from about 5 days of orbit of the first phase of Lunar Orbiter I. It was felt that only the C_{30} was sufficiently precise to warrant publication. However, subsequently Goudas, Kopal & Kopal (1966) have published the coefficients obtained by Michael's group transformed into equivalent height variations of a uniform density model.

Both solutions are given in table 1 in the form of fully normalized coefficients $\bar{C}_{lm}$, $\bar{S}_{lm}$: i.e. the coefficients in the spherical harmonic V_{lm} defined by

$$V_{lm} = \frac{GM}{r}\left(\frac{R}{r}\right)^{l} \bar{P}_{lm}(\sin\phi)\,[\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda], \tag{1}$$

where r, ϕ, λ are respectively the selenocentric radial distance, latitude, and longitude from the mean direction of the Earth; G is the gravitational constant; M, R are respectively the Moon's mass and radius; and $\bar{P}_{lm}$ is the normalized associated Legendre function:

$$\bar{P}_{lm}(\sin\phi) = \left[\frac{(l+m)!}{(l-m)!\,(2l+1)\,(2-\delta_{0m})}\right]^{\frac{1}{2}} \cos^{m}\phi \sum_{t=0}^{\frac{1}{2}(l-m)} \frac{(-1)^{t}(2l-2t)!}{2^{l}t!\,(l-t)!\,(l-m-2t)!}\sin^{l-m-2t}\phi. \tag{2}$$

In addition, table 1 gives the values obtained from the r.m.s. magnitude of the Earth's gravitational coefficients by the equal stress implication hypothesis (Kaula 1963, 1966):

$$\sigma_{l} = \sigma_{M}[\bar{C}_{lm}, \bar{S}_{lm}] = \left(\frac{g_{E}}{g_{M}}\right)^{2}\sigma_{E} = 6^{2}\,\frac{8\times10^{-6}}{l^{2}} = \pm\frac{2.9\times10^{-4}}{l^{2}}; \tag{3}$$

and from the Moon's orbital motion and librations. The values given in table 1 depend on Eckert's (1965) values from nodal motion:

$$g = \tfrac{3}{2}\,C/MR^{2} = 0.965\pm0.030; \tag{4}$$

and from perigee motion: $$f = \frac{C-B}{C-A} = 0.638\pm0.012, \tag{5}$$

which depend on, from the inclination 1° 32·2′ of the Moon's equator to the orbit,

$$\beta = \frac{C-A}{B} \approx \frac{C-A}{C} = 0.000629\pm0.000002. \tag{6}$$

Also used is Jeffreys's (1961) value from the forced libration in longitude:

$$\gamma = (B-A)/C = 0.0002049\pm0.0000009. \tag{7}$$

The uncertainties given in (4) and (5) correspond to $\pm 1{\cdot}0''$ (century)$^{-1}$ in $\dot{\Omega}$ and $\dot{\omega}$; those given in (6) and (7) are quoted from Jeffreys (1961). A, B, and C are the moments of inertia about the two equatorial and the polar axes of inertia respectively. Their relation to the gravitational coefficients is

$$\left.\begin{aligned}\bar{C}_{20} &= -\left(\tfrac{1}{5}\right)^{\frac{1}{2}}\frac{C-\frac{1}{2}(A+B)}{MR^2} = \left(\tfrac{1}{5}\right)^{\frac{1}{2}}\frac{g\beta(1+f)}{3},\\ \bar{C}_{22} &= \left(\tfrac{3}{5}\right)^{\frac{1}{2}}\frac{B-A}{2MR^2} = \left(\tfrac{3}{5}\right)^{\frac{1}{2}}\frac{3g\gamma}{4}.\end{aligned}\right\} \tag{8}$$

TABLE 1. THE COEFFICIENTS OF THE MOON'S GRAVITATIONAL FIELD

coefficient	Luna-10 (Akim 1966) $10^{-6}\times$	Lunar Orbiter I (Michael *et al.* 1966)* $10^{-6}\times$	lunar motion and librations (Eckert 1965) (Jeffreys 1961) $10^{-6}\times$	equal-stress hypothesis, σ_l $10^{-6}\times$
$\bar{C}_{20}$	−92·1	−93	−148	
$\bar{C}_{21}$	−12·1	−5·2	—	
$\bar{S}_{21}$	−2·8	10·0	—	±72
$\bar{C}_{22}$	21·7	37·7	51	
$\bar{S}_{22}$	−2·2	1·4	0	
$\bar{C}_{30}$	−13·9	37·0	—	
$\bar{C}_{31}$	52·6	3·0	—	
$\bar{S}_{31}$	16·5	16·8	—	
$\bar{C}_{32}$	34·6	9·4	—	±32
$\bar{S}_{32}$	−2·0	26·7	—	
$\bar{C}_{33}$	—	8·4	—	
$\bar{S}_{33}$	—	−4·7	—	
$\bar{C}_{40}$	11·1	21·7	—	
$\bar{C}_{41}$	—	−3·3	—	
$\bar{S}_{41}$	—	12·9	—	
$\bar{C}_{42}$	—	6·9	—	
$\bar{S}_{42}$	—	26·5	—	±18
$\bar{C}_{43}$	—	2·1	—	
$\bar{S}_{43}$	—	1·2	—	
$\bar{C}_{44}$	—	3·5	—	
$\bar{S}_{44}$	—	−12·1	—	

* As published by Goudas *et al.* (1966).

Eckert's (1965) reaffirmation (4) of the large value of g (first, to my knowledge, pointed out by Cook 1959) derived by comparing Brown's lunar theory with the observed secular motions is, of course, notorious for indicating a moment of inertia considerably higher than that of a homogeneous sphere. If we assume homogeneity, $g = 0{\cdot}60$, and $f = (\beta-\gamma)/\beta$, then

$$\left.\begin{aligned}\bar{C}_{20} &= -92\times10^{-6},\\ \bar{C}_{22} &= 32\times10^{-6}.\end{aligned}\right\} \tag{9}$$

These values are much closer to those given in table 1 by Akim (1966) and attributed by Goudas *et al.* (1966) to Michael *et al.* (1966). However, in view of the difficulty of separating $\bar{C}_{20}$ from $\bar{C}_{40}$ in the satellite motion, we must remain sceptical that the values of $\bar{C}_{20}$ given are indeed determined entirely from the satellite motion.

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Of indubitable significance is the smallness of the coefficients compared to the values extrapolated from the Earth: the Moon is appreciably closer to hydrostatic equilibrium than is the Earth. We can derive from table 1:

	Luna-10	Lunar Orbiter I	equal-stress hypothesis
$10^6\,\sigma_2$	± 15	± 27	± 72
$10^6\,\sigma_3$	30	19	32
$10^6\,\sigma_4$	11	13	18

Since the effects of noise, systematic error, and ill-conditioning are virtually always to exaggerate the magnitude of small parameters, these results at such an early stage are quite impressive.

Not so impressive is the poor agreement between the solutions, except for the second-degree coefficients for which there existed an *a priori* notion. The obvious explanation is distortion by omitted higher-degree coefficients, for which the obvious solution is a combination of data from satellites of different inclination. It is therefore to be hoped that an exchange of tracking data can be arranged. If it is not, then a combination of sorts can still be made by assuming that the value $\hat{C}_{lm}$ of a single coefficient absorbs all of the variation Δs_{lm} of the principal periodicity on which its determination depends, and writing an equation of the form

$$\frac{\partial s}{\partial C_{lm}}\hat{C}_{lm} = \Delta s_{lm} = \frac{\partial s}{\partial C_{lm}}C_{lm} + \frac{\partial s}{\partial C_{(l+2)m}}C_{(l+2)m}. \tag{10}$$

Pairing with a similar equation for a satellite of different inclination enables estimation of both C_{lm} and $C_{(l+2)m}$. The duration of data on which are based the coefficients reported by Akim (1966) is long enough that the dominant perturbations are long-periodic, as indicated by the figure in Akim's paper, so that the m-cycle per month essentially determines each C_{lm}. The duration covered by the Lunar Orbiter I data on which the results so far published were based is too short, however, to assure that the long-periodic variations were dominant in their determination.

As pointed out by Goudas *et al.* (1966), the estimate of the gravitational field by Michael *et al.* (1966) has appreciable resemblance to the variations of surface topography of the Moon. The global representation of the topography is derived from incomplete data by applying the 'mirror' assumption (Goudas 1966): i.e. replacing the harmonic analysis

$$\bar{C}_{lm} \quad \text{or} \quad \bar{S}_{lm} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} h(\phi,\lambda)\,\bar{Y}_{lm}(\phi,\lambda)\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\lambda \tag{11}$$

by the analysis

$$\bar{C}_{lm} \quad \text{or} \quad \bar{S}_{lm} = \frac{1}{4\pi}\Big[\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_0^{\pi} h(\phi,\lambda)\bar{Y}_{lm}(\phi,\lambda)\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\lambda + \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\int_0^{\pi} h(\phi,\pi-\lambda)\bar{Y}_{lm}(\phi,\lambda)\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\lambda\Big], \tag{12}$$

where h is the topography elevation and $\bar{Y}_{lm}$ is the normalized surface spherical harmonic. Because of this incompleteness, it seems more appropriate to make a comparison of the two types of data by converting the gravitational coefficients to the equivalent elevation coefficients for a uniform density Moon by using the formula

$$\left.\begin{matrix}\overline{C}_{lm}\,(h)\\ \overline{S}_{lm}\,(h)\end{matrix}\right\} = R^{(2l+1)/3}\left\{\begin{matrix}\overline{C}_{lm}\,(V)\\ \overline{S}_{lm}\,(V)\end{matrix}\right. \tag{13}$$

based on the transformation from surface density to potential (Jeffreys 1952). The comparison is given in table 2 with the topographic coefficients determined by Goudas (1966) from the elevations given by the Aeronautical Chart and Information Centre (A.C.I.C.) (Meyer & Ruffin 1965) and by Watts (1963).

The correlation coefficient between two solutions i, j is given by:

$$r_{ij} = \frac{\sum_{l,m} [\overline{C}_{lm,i}\overline{C}_{lm,j} + \overline{S}_{lm,i}\overline{S}_{lm,j}]}{[\sum_{l,m} (\overline{C}^2_{lm,i} + \overline{S}^2_{lm,j})]^{\frac{1}{2}}\,[\sum_{l,m} (\overline{C}^2_{lm,i} + \overline{S}^2_{lm,j})]^{\frac{1}{2}}}, \tag{14}$$

where the summation is over all coefficients common to the two solutions. The correlation coefficient of the topography with the Luna-10 determination is 0·33, and with the Lunar Orbiter I, 0·59. The latter is remarkably high, considering the imperfections that probably still exist in both data sets. It thus appears that the variations of the Moon's gravitational field are shallower in origin, as well as milder, than the Earth's.

Table 2. Comparison of variations in surface elevation calculated from gravity with those obtained from photographic measurements

	based on Luna-10 (km)	based on Lunar Orbiter I (km)	based on photographic data (km)
C_{20}	−0·267	−0·268	−0·166
S_{21}	0·008	0·029	0·318
C_{22}	0·062	0·109	0·465
C_{30}	−0·056	0·150	0·128
S_{31}	−0·067	0·068	0·111
C_{32}	0·140	0·038	−0·129
S_{33}	—	−0·019	−0·072
C_{40}	0·058	0·113	0·080
S_{41}	—	0·067	0·337
C_{42}	—	0·036	−0·353
S_{43}	—	0·006	0·268
C_{44}	—	0·018	0·393

The aforestated conclusions are based mainly on the tesseral harmonics of the gravitational field. These harmonics have been easier to distinguish because a larger part of their cycle of perturbations have taken place within the data spans available. The zonal harmonics are more difficult to distinguish because their effects are mainly secular. Hence we cannot yet decide whether the lunar satellites indicate a lower oblateness, J_2, than do the secular motions of the Moon's orbit. The closeness of the J_2's given in the brief papers of Akim (1966) and Goudas *et al.* (1966) to the J_2 for a homogeneous Moon, coupled with the giving of a J_4 value, makes it dubious that they are indeed independent determinations. We must wait for more detailed analyses before drawing any conclusions as to the Moon's moment of inertia.

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In addition to the aforedescribed solutions for the variations of the gravitational field, based mainly on Doppler tracking, some interesting results with ranging data have been recently obtained by Mulholland & Sjogren (1967). Residuals of ranging with respect to the lunar ephemeris as corrected by Eckert *et al.* (1966) are less than about ± 100 m for orbits fitted to 12 h or less of Doppler tracking.

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